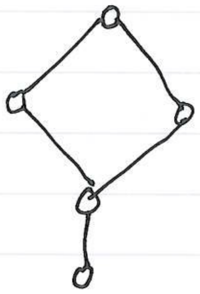
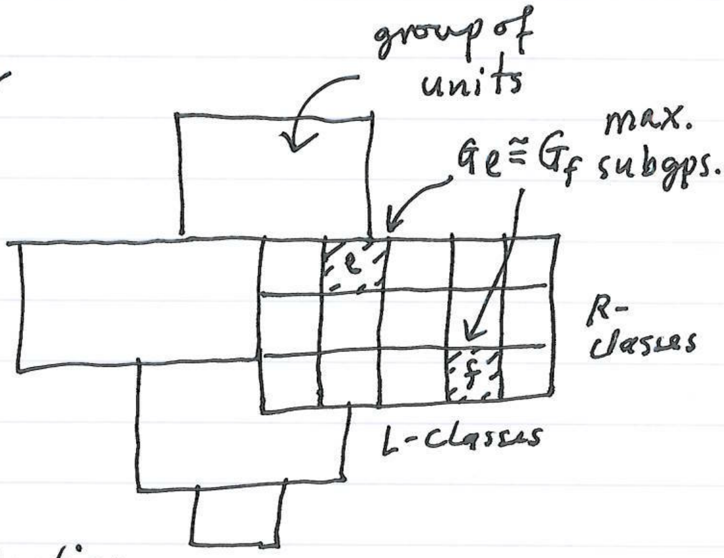
3. Reduction and induction

- Recall: S = finite regular monoid

poset of J-classes:

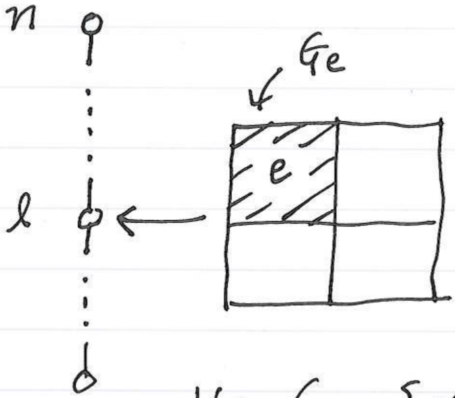




$$G_e\text{-representations} \underset{\text{reduction}}{\overset{\text{induction}}{\rightleftarrows}} S\text{-representations}$$

(1). Reduction (everyone else seems to say "restriction")

Eg: $S = I_n$, V = partial permutation rep.
(irreducible with $\dim V = n$)



$e = id_X : X \to X$, $|X| = \ell$

$G_e = \{\text{bijections } X \to X\}$
$\cong S_\ell$

$Ve \; (:= \{v.e \mid v \in V\}) = k$-space basis $\{v_i \mid i \in X\}$
($\Rightarrow \dim Ve = \ell$)

For $g \in G_e$ define

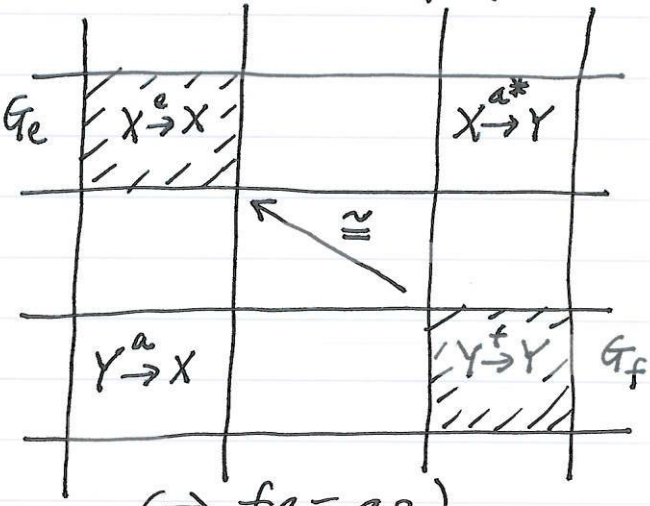
$$(v.e).g = v.(eg) \qquad (= v.(ge) = (v.g).e \in Ve)$$

$\Rightarrow Ve$ a G_e representation ($\cong$ permutation rep. of S_ℓ)

$G_f \cong S_Y \cong S_X \cong G_e$

via $h \mapsto a^* h a$

(${a^*}$ = inverse of a)



$(\Rightarrow fa = ae)$

$Vf \cong Ve$ via

$v\cdot f \mapsto v\cdot(fa) = v\cdot(ae)$
$= (v\cdot a)\cdot e$

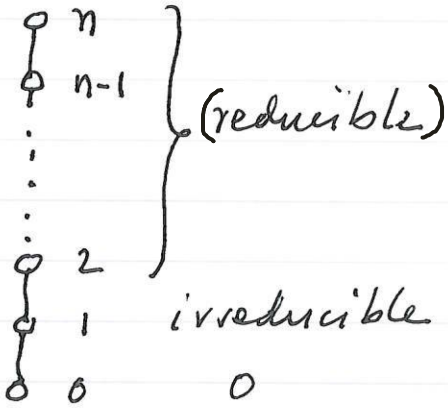
and

$$\begin{tikzcd} Vf \arrow[r, "(-)h"] \arrow[d, "\cong"'] & Vf \arrow[d, "\cong"] \\ Ve \arrow[r, "(-)a^*ha"] & Ve \end{tikzcd}$$

commutes

$\Rightarrow$ (upto $\cong$ of reps.) Ve does not depend on choice of idempotent in a J-class.

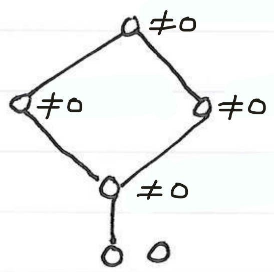
V partial permutation rep. for I_n (irreducible) $\longrightarrow$ Ve permutation rep. for S_ℓ $(0 \leq \ell \leq n)$



Eg: page 11a.

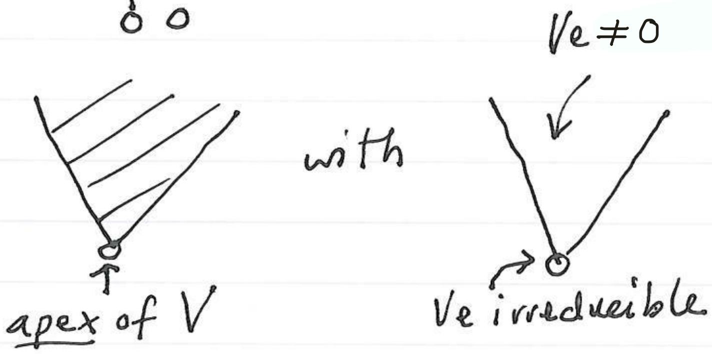
In general: S = finite regular monoid

V irreducible S-rep. $\xrightarrow{Ve}$



i.e: $Ve \neq 0$

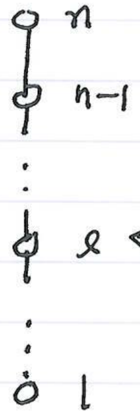
for $e \in J$-classes forming an interval



Eg: $S = T_n$, V = mapping rep. (reducible with $\dim V = n$) with basis $\{v_1, \ldots, v_n\}$

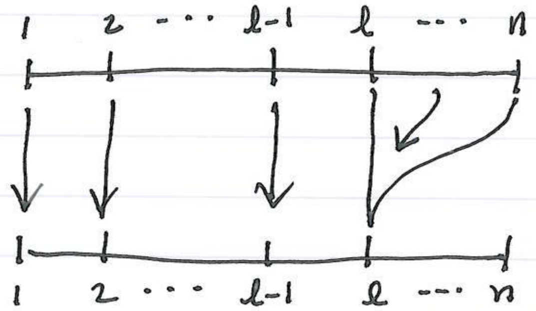
$W \subset V$ hyperplane $\sum x_i = 0$ (irred. with $\dim W = n-1$)

J-class poset:



all maps $[n] \to [n]$ with $|\mathrm{im}| = l$

$e =$



G_e = all bijections {fibres} $\to$ im(e)

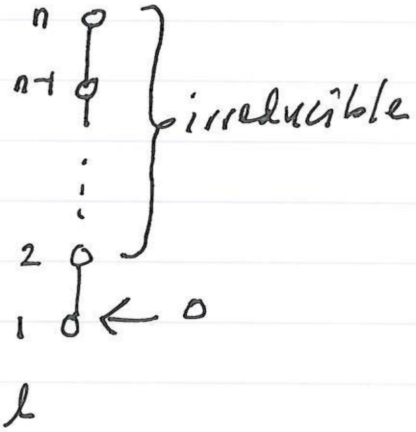
$\cong S_l$

V_e = k-space with basis $\{v_1, \ldots, v_l\}$

and $W_e \subset V_e$ the hyperplane $\sum x_i = 0$

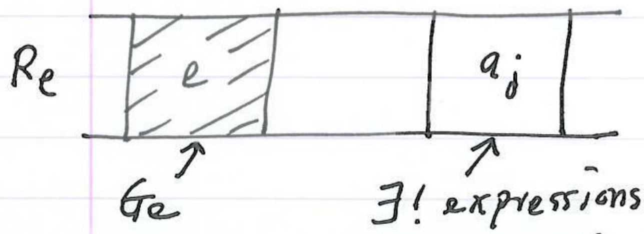
$V_e \cong$ permutation rep. of S_l

i.e: W irred. T_n-rep $\dim = n-1$ $\longrightarrow$ W_e S_l-rep $(1 \le l \le n)$ $\dim l-1$



Thus if V an irreducible S-representation and $e \in$ apex of V then $V \downarrow G_e := V_e$ an irreducible G_e-representation.

(2). <u>Induction</u> $\quad V$ a G_e-representation



Choose transversal $A = \{a_j\}$

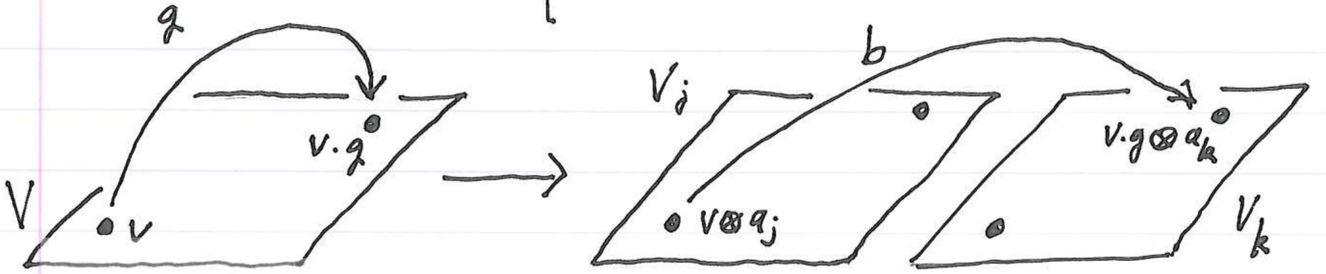
$g a_j \ (g \in G_e)$ $\quad \exists!$ expressions for H-classes in R_e with $e \in A$

For $a_j \in A$ let $V_j = \{ v \otimes a_j : v \in V \}$

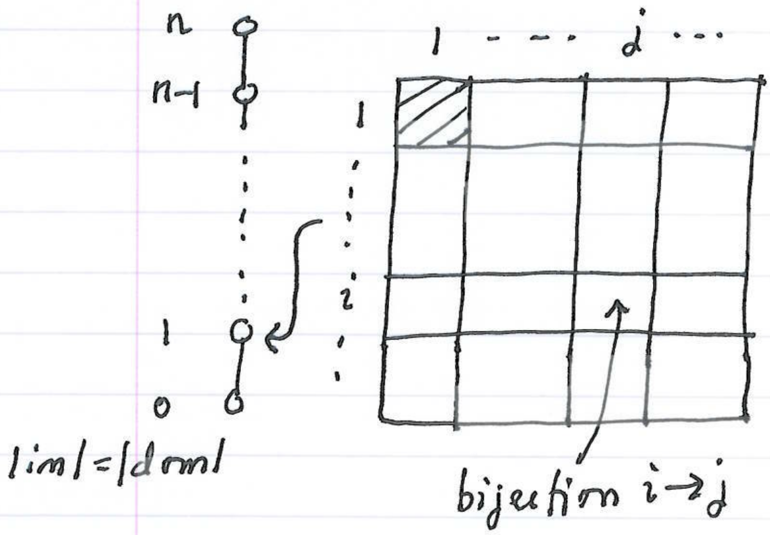
(a k-space $\cong V$ with $\lambda(v \otimes a_j) + \mu(u \otimes a_j) = (\lambda v + \mu u) \otimes a_j$)

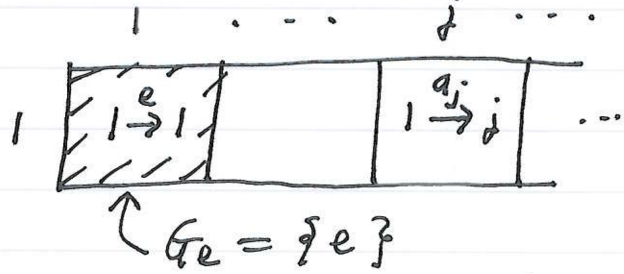
Define S-action on $\bigoplus_{a_j \in A} V_j$ by

$$(v \otimes a_j) \cdot b = \begin{cases} v \cdot g \otimes a_k, & a_j b \in R_e \Rightarrow a_j b = g a_k \\ 0, & a_j b \notin R_e. \end{cases}$$



Eg: $S = I_n$





$V =$ trivial G_e-rep. (irred.)

1-dim. with basis $\{v\}$ and $v \cdot e = v$

$\bigoplus_A V_j$ has basis $\{ v_j = v \otimes a_j \}$ with

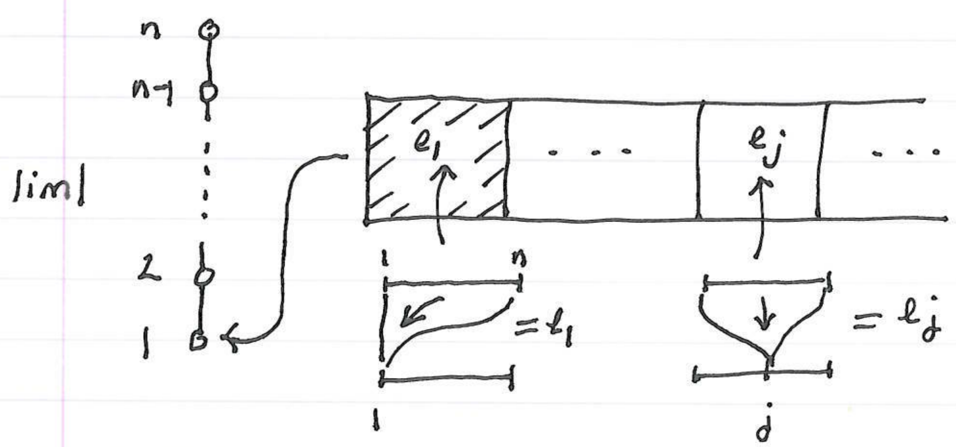
$$a_j b \in R_e \Leftrightarrow \mathrm{dom}(a_j b) = \{1\} \Leftrightarrow j \in \mathrm{dom}(b)$$

(in which case $a_j \cdot b = a_{jb}$)

$$\Rightarrow v_j \cdot b = \begin{cases} v \cdot e \otimes a_{jb} = v_{jb} , & j \in \mathrm{dom}(b) \\ 0 , & \text{else.} \end{cases}$$

the partial permutation rep. of I_n (irreducible)

Eg: $S = T_n$

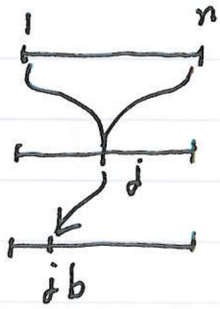


one R-class
n L-classes
(all idempotents)

$V =$ trivial rep. of G_{e_1}, basis $\{v\}$ (irreducible)

$\bigoplus_A V_j$ basis $\{v_j = v \otimes e_j\}$ with $e_j b = b_{jb}$

$\Rightarrow v_j \cdot b = v \cdot e \otimes e_{jb} = v_{jb}$



the mapping rep. of T_n, reducible

with sub-rep. $W = \{\sum \lambda_i v_i : \sum \lambda_i = 0\}$

notice: $L_{e_1} = \{e_1\}$ with $v_j \cdot e_1 = v_1$ for all j

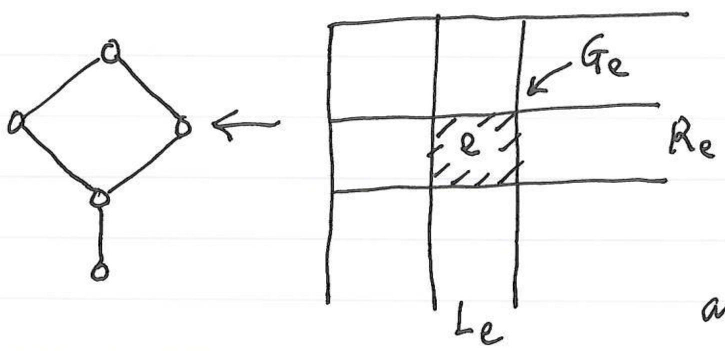
$$v = \sum \lambda_i v_i \text{ with } v \cdot e_1 = 0 \Leftrightarrow (\sum \lambda_i) v_1 = 0$$

$$\Leftrightarrow \sum \lambda_i = 0 \Leftrightarrow v \in W.$$

<u>in general</u>: V an S-representation and $U \subset V$ a subrepresentation $\Rightarrow$ quotient space V/U an S-representation via $(v+U)\cdot a = v \cdot a + U$.

$U \subset V$ <u>maximal</u> sub-representation $\overset{\text{def}^{\underline{n}}}{\Longleftrightarrow}$ given $U \subset \underset{\text{sub-rep.}}{W} \subset V$ we have $W = U$ or $W = V$.

Then U maximal $\Leftrightarrow$ V/U irreducible.



V an irreducible G_e-representation

$A = \{a_j\}$

and V_j as before

If $\mathrm{Ann}(L_e) = \{ v \in \bigoplus_A V_j : v.a = 0 \text{ for all } a \in L_e \}$

then $\mathrm{Ann}(L_e)$ (the unique) maximal subrepresentation

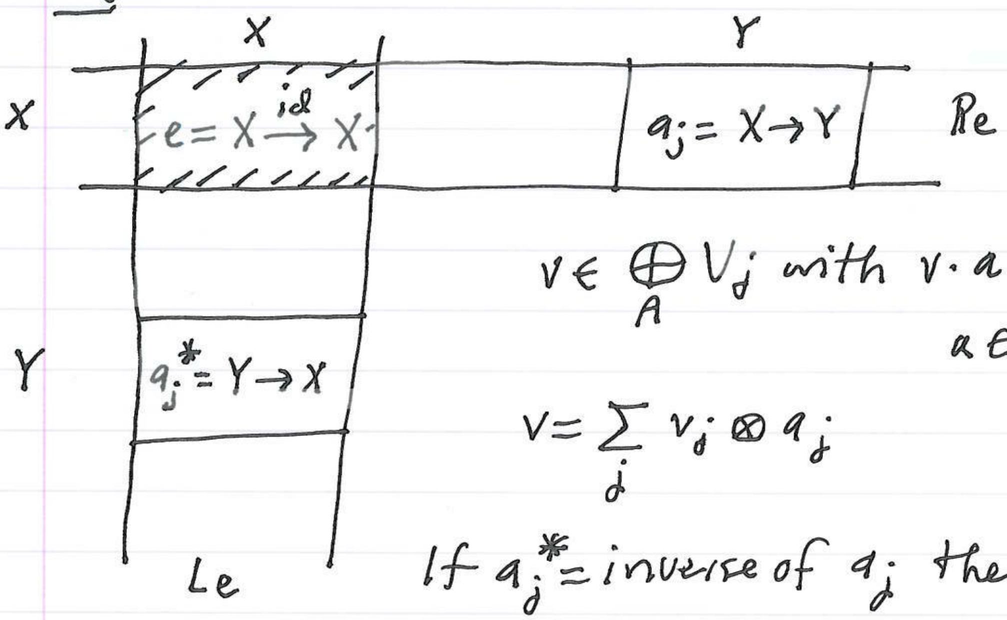
of $\bigoplus_A V_j$

$\Rightarrow V \uparrow S := \bigoplus_A V_j / \mathrm{Ann}(L_e)$ irreducible S-rep.

<u>Ex</u>: (upto $\cong$ of S-reps.) $V \uparrow S$ does not depend on

choice of $e \in J$; choice of transversal A.

<u>Eg</u>: $S = I_n$ and V a G_e-rep



$v \in \bigoplus_A V_j$ with $v.a = 0$ for all $a \in L_e$

$$v = \sum_j v_j \otimes a_j$$

If $a_j^* =$ inverse of a_j then

$$a_i a_j^* \in R_e \Leftrightarrow \mathrm{dom}(a_i a_j^*) = X$$

$$\Leftrightarrow \mathrm{im}\, a_i = \mathrm{dom}\, a_j^* = Y \Leftrightarrow i = j$$

so $0 = v \cdot a_j^* = (v_j \otimes a_j) \cdot a_j^* = v_j \otimes e$
$\qquad a_j a_j^* = e$

$\Rightarrow v_j = 0 \Rightarrow v_j \otimes a_j = 0$ (for all i) $\Rightarrow v = 0 \Rightarrow \mathrm{Ann}(L_e) = 0$

Ex: if S an inverse monoid and V a G_e-rep then

$\mathrm{Ann}(L_e) = 0$.

Eg: $S = T_n$, V = trivial rep. of G_e = trivial group

$V \uparrow S$ = mapping rep. $/ W$ (1-dimensional)

with basis $v_1 + W$ $\left(\begin{array}{l} v_i - v_j \in W \\ \Rightarrow v_i + W = v_j + W \end{array} \right)$

and $(v_1 + W) \cdot b = v_{1b} + W = v_1 + W$

= trivial rep. of T_n